A GENERALIZATION OF THE MIXING-LENGTH THEORY OF TURBULENT CONVECTION

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ABSTRACT

The mixing-length theory as currently employed is valid only when the mixing length is sufficiently small. The present work attempts to remove this limitation by writing a heat-transfer integral for convecting fluid elements. There then follows an integrodifferential equation for the mean temperature in a convecting medium. It is indicated how this equation may be used to include the effects of penetration into convectively stable regions from adjacent stable regions.

I. INTRODUCTION

The mixing-length theory of turbulence continues to be the principal theory in use for calculating convective heat fluxes in stars, in spite of severe doubts as to its validity and accuracy. It therefore seems reasonable to attempt to strengthen this theory as much as possible, though the need for a fundamentally better theory should not be overlooked or forgotten. The present note describes an attempt to remedy one deficiency of mixing-length theory as it is currently used, namely, the requirement that the mixing length be small. That this restriction is indeed inherent in the mixing-length theory in its present form will be shown in the next section.

The significance of the limitation to small mixing lengths may be seen from a combination of observational (Schwarzschild 1959) and theoretical studies (Vitense 1953). These permit one to say with fair certainty that the transition layer between the zones of radiative and convective equilibrium in the sun is of the order of a very few scale heights in thickness. Moreover, current ideas on stellar convection (Vitense 1953; Schwarzschild 1961) suggest that the mixing length is also of the order of the scale height. Hence the form of the mixing-length theory as now used is inaccurate in the transition layer in the same sense that the Eddington approximation in radiative-transfer theory fails near the top of a stellar atmosphere.

The idea underlying the present work is to write a transport equation for convecting masses of gas without appealing to the small mean-free-path approximation which usually enters the mixing-length theory. Our considerations will therefore produce an expression for convective heat flux which is an integral over the atmosphere analogous to the flux integral of radiative transfer. For the interior of the convection zone, the expression for the flux must reduce to that used in the customary mixing-length theory. We begin our discussion, therefore, by reviewing in the next two sections the equations for convective transfer currently in use. The subsequent sections will then treat the more general case.

II. THE MIXING-LENGTH APPROXIMATION

The starting point of the mixing-length theory for convective heat transfer is the exact equation

$$\Phi = \overline{\rho C_p w \theta}\,, \tag{1}$$

where Φ is the convective heat flux. We restrict ourselves to a plane-parallel atmosphere with the z-axis pointing upward. A horizontal overbar denotes averages over x and y, the horizontal co-ordinates; w is the vertical velocity; and θ is the fluctuation temperature. The total temperature may be written

$$T(x, y, z, t) = \bar{T}(z) + \theta(x, y, z, t) . \tag{2}$$

The mixing-length theory as used by Prandtl simplifies equation (1) in the following manner. First we assume that, at a given (x, y, z), the fluctuation temperature is due to the arrival of a convective element from the point (x, y, z_0). If we assume further that, at z_0, the convective element was at the local ambient temperature, we may write

$$T(x, y, z) = \bar{T}(z_0) + \frac{dT}{dz}(z - z_0) + \dots , \tag{3}$$

where dT/dz is the total temperature gradient as experienced by the convective element. We may also write the Taylor series for $\bar{T}(z)$:

$$\bar{T}(z) = \bar{T}(z_0) + \frac{d\bar{T}}{dz}(z - z_0) + \dots . \tag{4}$$

Hence, from equations (2), (3), and (4), we have

$$\theta(x, y, z) = \left(\frac{dT}{dz} - \frac{d\bar{T}}{dz}\right)(z - z_0) + \dots . \tag{5}$$

In calculating Φ with the Prandtl theory, one now retains only the leading term in equation (5) and identifies $2(z - z_0)$ with a characteristic mixing length, l. Equation (1) may then be written

$$\Phi = \tfrac{1}{2} \rho C_p w l \left(\frac{dT}{dz} - \frac{d\bar{T}}{dz}\right) \omega , \tag{6}$$

where each quantity on the right-hand side of equation (6) is given a characteristic value. The replacement of an average of a product by the product of averages is compensated by the phase factor ω. As in most mixing-length studies, we will here adopt $\omega = 1$, though a smaller value ($\sim\frac{1}{2}$) is undoubtedly called for when turbulent conductivity is large.

It is quite clear from this development that equation (6) will not be accurate unless l is in some sense a small quantity. Presumably, l should be small compared with the distance over which mean quantities, such as $\bar{T}$, vary appreciably. Unfortunately, in model-atmosphere calculations this condition is not always met. In particular, the important transition zone in the sun between the regions of radiative and convective equilibrium is of the order of l (Vitense 1953) in thickness.

It is the aim of the present paper to develop an expression for Φ which does not require l to be small. In doing this, we shall retain most of the other approximations of the current mixing-length theory. It will be useful, therefore, to have before us the equations of the theory as it is now used. In the next section we set these down.

III. THE MIXING-LENGTH FORMALISM

In order to specify a complete set of equations for the structure of a convective zone, we must supplement equation (6) with expressions for w and dT/dz. In the first astrophysical applications of mixing-length theory, dT/dz was taken to be the adiabatic gradient. However, Öpik (1950) and Mrs. Böhm (Vitense 1953) have shown that radiative energy loss by a convective element may significantly alter the efficiency of convection. Hence a more complicated expression for dT/dz is needed. The radiative corrections and the dynamical calculations of w have been made at the same level of precision as the derivations given in the preceding section. The resulting set of equations given by Mrs. Böhm is now standard in calculations of stellar structure. We shall summarize these equations here.

The mixing-length formalism is best summarized by the following equations:

$$\Phi = \tfrac{1}{2}\rho C_p w l \left(\frac{dT}{dz} - \frac{d\bar{T}}{dz}\right), \tag{7}$$

$$w = \tfrac{1}{2} l \sqrt{\left(\frac{g}{\bar{T}}\right)} \left(\frac{dT}{dz} - \frac{d\bar{T}}{dz}\right)^{1/2}, \tag{8}$$

and

$$\frac{(dT/dz) - (d\bar{T}/dz)}{(aT/dz)_{AD} - (dT/dz)} = \tfrac{1}{3}\frac{wl}{K}\rho C_p . \tag{9}$$

Here $(dT/dz)_{AD}$ is the adiabatic gradient, and K is the radiative conductivity. To these equations we must add the conservation equation,

$$\mathfrak{H} = \Phi + \mathfrak{F}, \tag{10}$$

where $\mathfrak{H}$ is the total flux and $\mathfrak{F}$ is the radiative flux; $\mathfrak{H}$ is a constant, and, for $\mathfrak{F}$, one normally enploys the expression

$$\mathfrak{F} = -K\frac{d\bar{T}}{dz}. \tag{11}$$

Equations (7)–(11) are then a complete set of equations from which we may calculate $\bar{T}(z)$, given $\mathfrak{H}$, an expression for l, an auxiliary expression for K, and the hydrostatic condition. Such calculations have been performed for a number of spectral types.

By algebraic substitutions, an equation for convective heat flux can be obtained; this is

$$(\rho C_p)^{-2/3}\left(\frac{g\alpha l^4}{16}\right)^{2/3}\Phi^{2/3} + \tfrac{3}{2}\kappa\left(\frac{g\alpha l^4}{16}\right)^{1/3}(\rho C_p)^{-1/3}\Phi^{1/3} - \frac{g\alpha\beta l^4}{16} = 0, \tag{12}$$

where

$$\kappa = \frac{K}{\rho C_p}, \tag{13}$$

$$\beta = -\left[\frac{d\bar{T}}{dz} - \left(\frac{dT}{dz}\right)_{AD}\right], \tag{14}$$

and

$$\alpha = \frac{1}{\bar{T}}. \tag{15}$$

Since equation (12) is quadratic in $\Phi^{1/3}$, we may solve for Φ. We find, on taking the appropriate root, that

$$\Phi = \rho C_p \frac{16(\tfrac{3}{4}\kappa)^3}{g\alpha l^4}\left[-1 + \left(1 + \frac{g\alpha\beta l^4}{9\kappa^2}\right)^{1/2}\right]^3. \tag{16}$$

Now expression (16) may be expressed more succinctly in terms of the growth rate of normal-mode analysis. In this analysis, with the neglect of fluctuating non-linear interactions, one finds a time dependence like $e^{\eta t}$ for each normal mode. If the normal modes are approximated by sinusoidal functions with horizontal wave number k_1 and total wave number k, we have (e.g., Ledoux, Schwarzschild, and Spiegel 1961)

$$\eta = -\frac{\kappa+\nu}{2}k^2\left\{1 - \left[1 - \frac{4\nu\kappa}{(\nu+\kappa)^2} + \frac{g\alpha\beta}{(\kappa+\nu)^2}\frac{4k_1^2}{k^6}\right]^{1/2}\right\}. \tag{17}$$

Here β is the constant linear gradient for the static state and ν is the kinematic viscosity. For astrophysical convection, $\nu \ll \kappa$, and we have

$$\eta = \frac{\kappa k^2}{2}\left[-1+\left(1+\frac{g\alpha\beta}{\kappa^2}\,\frac{4k_1^2}{k^6}\right)^{1/2}\right]. \tag{18}$$

Within the spirit of mixing-length theory we shall set

$$k = \frac{\pi}{l}, \tag{19}$$

adopt the shape of the most unstable modes, i.e.,

$$k_1^2 = \tfrac{1}{2}k^2, \tag{20}$$

and assign local values to β. Then we find

$$\eta = \tfrac{1}{2}\,\frac{\kappa}{l^2}\left[-1+\left(1+2\,\frac{g\alpha\beta l^4}{\pi^4 k^2}\right)^{1/2}\right]. \tag{21}$$

We note that the expression for η is very much like part of expression (16) for Φ. There are some differences in factors of π, and we shall here adopt the numerical factors as they appear in equation (21). This is not a fundamental matter, since we have no secure knowledge of l, and the uncertainties are at least as great as factors of π. We may write, finally,

$$\Phi = \tfrac{27}{8}\pi^4\,\frac{\rho C_p \eta^3 l^2}{g\alpha}. \tag{22}$$

This general form for Φ could be derived also by dimensional arguments if l and η^{-1} are adopted as the characteristic length and time. It may be of interest to recall here that in the Boussinesq approximation convection can occur in a layer with free boundaries whenever the Rayleigh number exceeds the value

$$R_c = \tfrac{27}{4}\pi^4. \tag{23}$$

IV. THE CONVECTIVE HEAT FLUX

In this section we shall derive an equation for Φ, the convective heat flux, which does not require that l be small. We shall retain the notion of convective elements and associate with them a mean-free path, or mixing length, l. We begin by describing the aggregate of convective elements by a distribution function $\phi(x_i, v_i, t)$, where x_i and v_i are vector position and velocity and ϕ is a number density in (x_i, v_i)-space. In the usual way we may derive the conservation equation,

$$\frac{\partial\phi}{\partial t}+\frac{\partial}{\partial x_j}(v_j\phi)+\frac{\partial}{\partial v_j}(v'_j\phi) = q-\frac{v\phi}{l}, \tag{24}$$

where q is a source term and the term $v\phi/l$ describes the disappearance of convective elements at the ends of their paths. The quantity v is the speed associated with v_i and the prime denotes differentiation with respect to t.

To understand equation (24) we must recall that in mixing-length theory it is assumed that at each point in the atmosphere there is a unique (or at least a well-defined mean) scale of the convecting elements. To each position there is also assigned a fixed value of v. The change in the properties of a convecting element as it travels in the buoyancy and pressure fields is described by the driving term $\partial(v'_j\phi)/\partial v_j$. The source term, q, represents the initial creation of convective elements either because of statistical fluctuations in the medium or by non-linear effects, which we shall not discuss explicitly.

In the present approach we shall combine the driving term with the source term and discuss their determination in the next section.

For the case of interest here, the steady-state plane-parallel case, equation (24) then becomes

$$\mu \frac{d(v\phi)}{dz} + \frac{v\phi}{l} = q_1, \tag{25}$$

where

$$q_1 = q - \frac{\partial}{\partial v_j}(v'_j\phi) \tag{26}$$

and μ is the direction cosine with respect to z.

Equation (25) resembles the equation for radiative transfer in a gray medium. This is true because we have assumed a unique l and v describing convective elements at each height in the atmosphere.

Let us now introduce the variables ψ and σ defined by

$$\psi = v\phi \tag{27}$$

and

$$d\sigma = -\frac{dz}{l}. \tag{28}$$

Equation (25) then transforms into

$$\mu \frac{d\psi}{d\sigma} = \psi - Q, \tag{29}$$

where

$$Q = q_1 l. \tag{30}$$

The formal solution for equation (29) is (e.g., Chandrasekhar 1950)

$$\psi(\sigma, \mu) = \begin{cases} \int_\sigma^\infty Q(s)\exp\left(\frac{\sigma - s}{\mu}\right)\frac{ds}{\mu}, & \mu > 0 \\ -\int_0^\sigma Q(s)\exp\left(\frac{\sigma - s}{\mu}\right)\frac{ds}{\mu}, & \mu < 0. \end{cases} \tag{31}$$

In writing this solution we have assumed that

$$\psi(0, \mu) = 0 \qquad \text{for } \mu < 0. \tag{32}$$

Now, let e_+ be the energy excess of an outward-moving convective element. The outward convective heat flux then is

$$\Phi_+(\sigma) = 2\pi \int_0^1 e_+ \mu\psi d\mu. \tag{33}$$

Transformations familiar in transfer theory permit us to rewrite this as

$$\Phi_+(\sigma) = 2\pi \int_\sigma^\infty e_+ Q(s) E_2(s - \sigma) ds, \tag{34}$$

where E_2 is the second exponential integral. Similarly if e_- is the energy excess of inward-moving masses, the back-flux is

$$\Phi_-(\sigma) = -2\pi \int_0^\sigma e_- Q(s) E_2(\sigma - s) ds. \tag{35}$$

The net flux is given by

$$\Phi = \Phi_+ + \Phi_- . \tag{36}$$

In the context of mixing-length theory we must naturally set

$$e_- = -e_+ , \tag{37}$$

where e_+ is presumably positive. This means that the downward flux of gas masses produces an upward flow of heat. We find, then, that

$$\Phi = 2\pi \int_0^\infty \mathfrak{Q}(s) E_2(|s-\sigma|)\, ds \tag{38}$$

where

$$\mathfrak{Q} = e_+ Q . \tag{39}$$

In order to complete the specification of heat flux we must give an expression for $\mathfrak{Q}$, and we shall take up this question in the next section. At this point, however, we may remark that equation (38) is the formal generalization that we should like to propose for computing convective heat fluxes in regions where the mean properties vary rapidly. Though we have used most of the approximations of mixing-length theory in arriving at this equation, we have nowhere required l to be small.

V. AN EXPRESSION FOR $\mathfrak{Q}$

To derive from first principles an expression for the "source" function, $\mathfrak{Q}$, we would have to turn to the dynamics of the driving term in equation (24). The calculations which are called for are similar to those which underlie the derivation of equation (8). Indeed, we can do no better than this in the context of mixing-length theory, since we are necessarily in ignorance of the details of turbulent dynamics. These difficulties lead us to attempt to by-pass the dynamical equations by connecting our present approximation directly to the usual mixing-length theory. We may do this by recognizing that our present theory should be equivalent to the Prandtl-type theory when the mixing length is sufficiently small.

The equivalence of the generalized theory with the Prandtl-Vitense theory must appear when we consider the convective flux deep in the unstable layer where the mixing length is effectively small. There is here a close analogy with radiative transfer, which reduces to a short photon mean-free-path theory (Eddington approximation) at large optical depth. Our intention here is to take advantage of this reduction in suggesting an expression for $\mathfrak{Q}$.

In the appendix we shall derive an asymptotic expression for Φ at very large σ. The result of this calculation leads us to

$$\mathfrak{Q} = \frac{1}{2\pi}\Phi + 0(e^{-\sigma}) \tag{40}$$

for large σ. In deriving expression (40), it is necessary to use the fact that, at large σ, there is convective equilibrium, so that $\Phi \cong \mathfrak{H}$.

It is interesting to compare equation (40) with the corresponding asymptotic limit of radiative-transfer theory. There the source function is asymptotically linear in optical depth for large optical depth. The difference is due to the property of convective heat flux that downward-moving elements convect heat upward.

Consistent with the reasoning we have followed here, we may combine equations (40) and (22) to obtain

$$\mathfrak{Q} = \frac{27\pi^3}{16} \frac{\rho C_p \eta^3 l^2}{g a}, \tag{41}$$

which is valid for large σ. It must be stressed, however, that we have not reduced the right-hand side of equation (41) to its asymptotic limit for large σ. In this limit the Péclet number, wl/κ, becomes large, and turbulent conductivity dominates radiative conductivity. The general form expressed in equation (41), however, includes both radiative and turbulent effects.

Our problem now is to suggest a form for $\mathfrak{Q}$ that is valid at small σ. To do this, we first try to generalize our expression for η. As we have defined it, η depends on the local value of the gradient, β. This local property enters the usual mixing-length theory when the buoyancy force on a convective element is approximated by its value at the center of the element. However, when l is not small, such evaluation is not adequate, and the value of β and of η must be representative of the entire convective element rather than just the center.

To generalize the expression for η to the case of large l, we replace β in equation (21) by an appropriate average, β_m. Variational calculations suggest that β_m may be represented by

$$\beta_m(z) = \frac{\int_{z-l/2}^{z+l/2} w^2(z')\beta(z')dz'}{\int_{z-l/2}^{z+l/2} w^2(z')dz'} . \tag{42}$$

With this replacement, equation (21) for η is (at least approximately) applicable to the case of large l.

Returning, then, to the choice of $\mathfrak{Q}$, we note that expression (41) is the form which follows from dimensional reasoning if the new η^{-1} is the unit of time and l is the unit of length. Hence the simplest and most natural generalization of $\mathfrak{Q}$ to the case of large l (or small σ) is expression (41), with η given its new meaning through equation (42). For calculational purposes, the choice

$$w(z') = \sin\frac{\pi}{l}(z - z' + \tfrac{1}{2}l) \tag{43}$$

is then a convenient and reasonably accurate form for use in evaluating β_m by expression (42).

VI. THE EQUATION FOR $\bar{T}$

We may now combine equations (10), (11), and (38) to obtain

$$2\pi\int_0^\infty E_2(|s-\sigma|)\mathfrak{Q}(s)ds - K\frac{d\bar{T}}{dz} = \mathfrak{H} , \tag{44}$$

which we must solve for $\bar{T}$. For $\mathfrak{Q}$ we have expression (41), which is in turn supplemented by equations (21) and (42). The total flux, $\mathfrak{H}$, is also given, being related to the effective temperature by

$$\mathfrak{H} = \sigma T_e^4 . \tag{45}$$

The mixing length, l, on the other hand, remains highly uncertain, and we shall not offer any suggestions here as to its possible values.

It remains only to specify the nature of the boundary of the convection zone. This boundary is not sharp but represents a gradual transition between the radiative and convective zones. In her treatment of the problem, Mrs. Böhm (e.g., Vitense 1953) takes the depth at which $d\bar{T}/dz = dT/dz$ as the edge of the convection zone. This choice neglects convective transfer by elements which penetrate up into the region of stable temperature gradient. A similar choice is possible here, but it seems preferable to try to calculate the heat transfer by the penetrative convection.

Thus we suggest that equation (44) be used as long as any convective transfer results from the integral expression. One complication is that when β_m becomes negative, η in the expression for $\mathfrak{Q}$ will become complex. It will, however, have a negative real part. The procedure should then be to take only the real part of η as a measure of the damping of the convective motions. Hence, when η becomes negative, so will $\mathfrak{Q}$, and the integral expression in equation (43) will approach zero as higher levels in the atmosphere are approached. At the level where the integral just vanishes, the gradient is radiative, and this marks the edge of the convective zone. At higher levels the equations of radiative equilibrium obtain. (Similar considerations apply to the lower boundary of the convection zone.)

VII. CONCLUDING REMARKS

In assessing the usefulness of our proposed generalization of the mixing-length theory, we must review briefly the basis of the theory itself. To begin with, it should be stressed that the word *theory* in this context is perhaps a misnomer. Many authors have emphasized that the mixing length is to be taken as an empirical quantity which serves as a parameter describing turbulence. The equations presented in the theory are then just a convenient formalism devised for parameterizing the turbulent processes. That the equations are based on approximate physical pictures is necessary to make the empirical description flexible and accurate enough to accommodate the complexities of turbulent dynamics.

The motivation of the present work, then, is to add further structure to the mixing-length formalism in the hope that the improved version will make a better representation of turbulent convection possible. It seems plausible that this end may be achieved by building the formalism on a more accurate physical model than has been used previously. The present picture, however, must still be regarded as a formalism for describing stellar convection rather than a complete theory, though the hope is that it may make possible a more accurate description than is possible with the short-mixing-length approximation.

Of course, the necessities of astrophysical application perhaps make this view of the mixing-length theory too restrictive. There is scant hope of an appropriate empirical determination of the mixing-length parameter, and it is necessary to make plausible estimates of the mixing length in order to calculate models. The inherent uncertainty of this procedure is clear from the known sensitivity of the models to the choice of mixing length. However, alternative approaches to the problem (e.g., Malkus 1954; Ledoux, Schwarzschild, and Spiegel 1961; Spiegel 1962) equally have their difficulties, and this has encouraged the author to attempt the present improvement of the mixing-length theory.

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APPENDIX

Asymptotic Form for Φ

We wish here to consider the asymptotic limit of expression (38) for Φ as σ tends to infinity. We begin by rewriting equation (38) as

$$\Phi(\sigma) = 2\pi \int_{\sigma}^{\infty} \mathfrak{Q}(s) E_2(s-\sigma)\, ds + 2\pi \int_{0}^{\sigma} \mathfrak{Q}(s) E_2(\sigma - s)\, ds \,, \tag{46}$$

With simple changes in integration variables, this becomes

$$\Phi(\sigma) = 2\pi\int_0^\infty \mathfrak{Q}(y+\sigma)E_2(y)\,dy + 2\pi\int_0^\sigma \mathfrak{Q}(\sigma-y)E_2(y)\,dy. \tag{47}$$

The function $E_2(y)$ is peaked about $y = 0$, and, to approximate the integrals of equation (47), we may expand $\mathfrak{Q}$ in Taylor series about $y = 0$. In this we take for granted absolute convergence in the domain of the convecting layer.

The series we need are

$$\left.\begin{aligned} \mathfrak{Q}(y+\sigma) &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathfrak{Q}^{(n)}(\sigma)\,y^n, \\ \mathfrak{Q}(\sigma-y) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathfrak{Q}^{(n)}(\sigma)\,y^n, \end{aligned}\right\} \tag{48}$$

where $\mathfrak{Q}^{(n)}$ denotes the nth derivative of $\mathfrak{Q}$. If we introduce these series into equation (47), we obtain, after some elementary operations,

$$\begin{aligned} \frac{1}{2\pi}\Phi(\sigma) = &\sum_{n=0}^{\infty}\frac{1}{n+1}\mathfrak{Q}^{(2n)}(\sigma) \\ &+E_2(\sigma)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\mathfrak{Q}^{(n)}(\sigma)\,\sigma^n \\ &+E_1(\sigma)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)!}\mathfrak{Q}^{(n)}(\sigma)\,\sigma^n \\ &-\tfrac{1}{2}\sigma\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)!}\mathfrak{Q}^{(n)}(\sigma)\,W_{(n+1)/2,\,(n+2)/2}(\sigma)\,\sigma^{n+1/2}, \end{aligned} \tag{49}$$

where $W_{\alpha,\beta}$ is the Whittaker function (Magnus and Oberhettinger 1954). If we consider large values of σ, equation (49) reduces to

$$\frac{1}{2\pi}\Phi(\sigma) = \sum_{n=0}^{\infty}\frac{1}{n+1}\mathfrak{Q}^{(2n)}(\sigma) + 0(e^{-\sigma}), \tag{50}$$

since E_1, E_2, and $W_{\alpha,\beta}$ are exponentially small.

Equation (50) is a differential equation whose solution gives the relation between $\mathfrak{Q}$ and Φ for large σ. The homogeneous solution is a linear combination of functions of exponential order. Those of positive exponential order must be discarded for realizability, those of negative order are negligible at large σ. We are left with only the particular solution to determine, but first we introduce a further approximation. We shall assume that, for large σ, the convective flux dominates the radiative flux, so that Φ may be considered constant at large σ. Equation (50) then leads us to

$$\mathfrak{Q} = \frac{1}{2\pi}\Phi + 0(e^{-\sigma}). \tag{51}$$

Equation (51), which asserts that $\mathfrak{Q}$ approaches a constant at large σ, differs from the corresponding result of radiative-transfer theory. There one derives that the source function in LTE becomes linear in optical depth at large optical depth. The reason for the difference is that downward-descending gas masses produce an upward flux of heat in mixing-length theory, while inward- and outward-directed beams of radiation work at cross-purposes, as far as flux is concerned.

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